Where $\bar{\rho} = \bar{\rho}(z)$ is a steady reference-state density. In addition, both approximations involve simplifications in the momentum equations (see Durran 1999, pp. 20--26). Another type of approximation in this class is the pseudo-incompressible approximation described by Durran (1989).

\subsection{Shallow-fluid equations}

The shallow water equations, also known as the Saint-Venant equations, are a set of hyperbolic partial differential equations that describe the flow of a shallow fluid like water over a surface. These equations are industry stnadard, used to model flood wave propagation and inundation over floodplains and river channels.

The key assumptions made in deriving the shallow water equations are:

\begin{itemize}

\item The pressure is hydrostatic and negligible

\item The water is shallow compared to the coverage

\item The velocity is in the horizontal plane

\end{itemize}

Where:

\begin{itemize}

\item h = water depth

\item u, v = x, y velocity components

\item g = gravitational acceleration

\item $C\_f$ = friction coefficient

\end{itemize}

These equations account for the major forces acting on the shallow water flow - inertia, pressure, gravity, and friction. Solving them numerically allows prediction of the water depth and velocity over a 2D domain like a floodplain. For 1D applications over river channels, the shallow water equations reduce to the 1D St. Venant equations. The shallow water equations are challenging to solve numerically due to their non-linear, hyperbolic nature. Various numerical schemes have been developed, including finite difference, finite volume, and discontinuous Galerkin methods.

The shallow-fluid equations, sometimes called the shallow-water equations, can serve as the basis for a simple model that can be used to illustrate and evaluate the properties of numerical schemes. Inertia--gravity, advective, and Rossby waves can be represented. Not only is such a valuable model for gaining experience with numerical methods, but the fact that the equations represent much of the horizontal dynamics of full baroclinic models also makes it a valuable tool for testing numerical methods in a simple framework.

For a fluid assumed to be autobarotropic (barotropic by definition, not by the prevailing atmospheric conditions), homogeneous, incompressible, hydrostatic, and inviscid, we use the following parameters:

\begin{itemize}

\item Velocity Components:

\begin{itemize}

\item $u$ - Velocity in x-direction

\item $v$ - Velocity in y-direction

\item $w$ - Velocity in z-direction

\end{itemize}

\item Physical Parameters:

\begin{itemize}

\item $\rho$ - Fluid density

\item $p$ - Pressure

\item $g$ - Acceleration due to gravity

\end{itemize}

\item Coriolis Parameter:

\begin{itemize}

\item $f$ - Coriolis parameter (representing Earth's rotation effects)

\end{itemize}

\end{itemize}

The governing equations are then expressed as:

\begin{align}

\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + \frac{1}{\rho}\frac{\partial p}{\partial x} &= 0 \label{eq:35} \\

\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu + \frac{1}{\rho}\frac{\partial p}{\partial y} &= 0 \label{eq:36} \\

\frac{\partial p}{\partial z} &= -\rho g \label{eq:37}

\end{align}

Now, incompressibility and homogeneity imply:

\begin{equation}

\frac{d\rho}{dt} = 0, \text{ for } \rho = \rho\_0 \text{ a constant}

\label{eq:39}

\end{equation}

And:

\begin{equation}

\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0

\label{eq:40}

\end{equation}

The hydrostatic equation can thus be written:

\begin{equation}

\frac{\partial p}{\partial z} = -\rho\_0 g

\label{eq:41}

\end{equation}

For complete details on the shallow-fluid equations and their numerical solution, see Kinnmark (1985), Pedlosky (1987), Durran (1999), and McWilliams (2006).

\subsection{Numerical methods}

\section{Numerical Solutions for Shallow Water Equations}

The shallow water equations are a set of hyperbolic partial differential equations that describe fluid flow under the assumption that the horizontal length scale is much greater than the vertical scale. These equations are widely used to model river flows, coastal areas, and the atmosphere. The following sections outline key numerical methods and references for solving these equations.

\subsection{Numerical Methods}

Several numerical methods are employed to solve the shallow water equations:

\begin{enumerate}

\item Finite Difference Methods:

\begin{itemize}

\item Lax-Friedrichs scheme

\item Lax-Wendroff scheme

\item Leap-Frog scheme

\end{itemize}

\item Finite Volume Methods:

\begin{itemize}

\item Roe's approximate Riemann solver

\item HLL-type solvers

\item Flux vector splitting

\item Flux difference splitting

\end{itemize}

\item Finite Element Methods

\item Spectral Methods

\item High-Order Methods:

\begin{itemize}

\item High-order finite difference schemes

\item SBP-SAT method (Summation-By-Parts Simultaneous Approximation Term)

\end{itemize}

\end{enumerate}